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Nonlinear hydromagnetic waves in a thermally stratified cylindrical fluid. Part 1. Exact translationally and axially symmetric solutions

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The propagation of nonlinear hydromagnetic waves in a highly conducting, self-gravitating fluid in a cylindrical geometry, subject to the convective forces produced by a radial temperature gradient, is treated in a Boussinesq approximation. Exact wave solutions of the nonlinear magnetohydrodynamic equations (in the Boussinesq approximation) in the presence of convective forces are obtained for the case when the physical quantities are independent of the axial coordinate or the azimuthal angle in the cylindrical coordinates. The solutions represent waves propagating in the azimuthal or axial direction under the influence of the helical magnetic and velocity fields and the convective forces. The solutions may be applicable to the hydromagnetic waves in the Earth's fluid core and the solar convection zone.

1. Introduction

Hydromagnetic (or Alfvén) waves are highly significant in conducting fluids in a magnetic field. It is well known that the magnetohydrodynamic (MHD) equations admit solutions representing Alfvén waves even in an incompressible inviscid fluid. The existence of such a wave of small amplitude was first predicted by Alfvén in 1942 (Alfvén 1942, 1950). The existence of non-dispersive Alfvén waves of arbitrary amplitude was subsequently suggested by Walén (1944), who concluded that an Alfvén wave of any form, frequency and amplitude could exist in a uniform magnetic field. Recently, Hamabata (1990*a*) found a class of exact nonlinear wave solutions to the MHD equations with large amplitude propagating in a straight but non-uniform magnetic field with constant or non-uniform velocity. Some exact nonlinear wave solutions to the MHD equations in a cylindrical geometry, which represent waves propagating in the azimuthal or axial direction or propagating helically on the cylindrical surfaces under the influence of the helical magnetic and velocity fields whose strength varies with radius, were also obtained by Namikawa & Hamabata (1987, 1988), Hamabata & Namikawa (1989) and Hamabata (1990*b*, *c*).

It is widely recognized that the strong azimuthal (toroidal) magnetic field is confined to the Earth's core or the solar convection zone, where it is generated from the weak poloidal magnetic field by the differential rotation, and hence the magnetic field is generally helical. Studies of hydromagnetic waves under the influence of the helical magnetic field, the sheared velocity and the convective forces are of some geophysical and astrophysical interest and may be a useful stepping-stone in the solar dynamo or geodynamo problem. In this connection, Parker (1984) found exact nonlinear Alfvén wave solutions with large amplitude but restricted form propagating along a uniform horizontal magnetic field in a highly conducting incompressible fluid subject to the convective forces produced by a uniform vertical temperature gradient within a Boussinesq approximation. As stated above, however, an important property of the solar and the Earth's magnetic field is the curvature of the field lines. Thus, a local analysis of hydromagnetic waves which uses a plane layer with a uniform magnetic field does not model some of essential physics in the Earth's core and the solar convection zone, and one hence is faced with the difficulty of solving the global wave propagation problem. A spherical geometry is often too difficult to work with, so a useful compromise is to consider an infinite cylinder with a helical magnetic field and a shear flow. Hence in this paper we wish to determine whether there are exact wave solutions to the dynamical equations (in the Boussinesq approximation) for a fluid in a cylindrical geometry subject to the convective forces produced by a radial temperature gradient, by extending the analysis by Hamabata (1990*b*) to include the effect of stratification. As we shall see, there is a class of exact wave solutions of arbitrary amplitude but restricted form propagating in the azimuthal direction and in the axial direction under the influence of the helical magnetic and velocity fields which may be of some physical interest in the Earth's core as well as in the solar convection zone.

We consider the motion of an electrically conducting self-gravitating fluid in an infinite cylindrical annulus $r_i < r < r_0$ with respect to a cylindrical coordinate system (r, ϕ, z) (with unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\phi}}$ and $\hat{\boldsymbol{z}}$). We assume that the Boussinesq approximation is valid and that effects of thermal, viscous, and magnetic dissipation may be neglected. The equations describing the velocity, magnetic field and temperature variations are

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V}\cdot\nabla\boldsymbol{V} = -\nabla P + \boldsymbol{H}\cdot\nabla\boldsymbol{H} - \alpha T\nabla\Phi, \tag{1}$$

$$\frac{\partial \boldsymbol{H}}{\partial t} = \nabla\times(\boldsymbol{V}\times\boldsymbol{H}), \tag{2}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{V}\cdot\nabla T = 0, \tag{3}$$

$$\nabla\cdot\boldsymbol{V} = 0, \tag{4}$$

$$\nabla\cdot\boldsymbol{H} = 0, \tag{5}$$

where $\boldsymbol{V}$ is the fluid velocity, $\boldsymbol{H} = \boldsymbol{B}/(4\pi\rho)^{\frac{1}{2}}$ is the equivalent Alfvén velocity as a measure of the magnetic field $\boldsymbol{B}$, $P = p/\rho - \Phi + \frac{1}{2}H^2$ is the modified pressure including the magnetic pressure as well as the undisturbed gravitational potential $\Phi(r)$, and T is the temperature. Here the fluid is assumed to be subject to a linear thermal expansion with coefficient α, so that the deviation from the constant mean density ρ is $\Delta\rho = -\alpha\rho T$. The gravitational potential is determined by

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left[r\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right] = -4\pi G\rho, \tag{6}$$

where G is the constant of gravitation.

We are interested in deformations of a system whose undisturbed state involves the helical magnetic field $\boldsymbol{H}_0(r)$ and the helical velocity field $\boldsymbol{V}_0(r)$ which are given by

$$\boldsymbol{H}_0 = r\tau(r)\,\hat{\boldsymbol{\phi}} + H(r)\,\hat{\boldsymbol{z}}, \tag{7}$$

$$\boldsymbol{V}_0 = r(\Omega_0 + \zeta(r))\,\hat{\boldsymbol{\phi}} + V(r)\,\hat{\boldsymbol{z}}, \tag{8}$$

and also involves the temperature distribution $T_0(r)$ whose radial gradient may be maintained by a distribution of heat sources. Here τ is the Alfvén frequency of the azimuthal component of the magnetic field, Ω_0 is a constant, ζ is the velocity shear of the azimuthal flow component. We assume that $\zeta(r_0) = V(r_0) = 0$, $\tau(r_0) = \tau_0$ and $H(r_0) = H_0$ with τ_0 and H_0 constants, that is, the flow is zonal with velocity $r_0 \Omega_0$ at the outer boundary $r = r_0$.

In §2 we consider the case when the physical quantities are independent of z. In §3 we also consider the case when the physical quantities are independent of ϕ. Simple solutions to the reduced equations derived in §§2 and 3 satisfying the above conditions are obtained to represent the examples of translationally and axially symmetric hydromagnetic wave motions in a thermally stratified cylindrical fluid in §4.

2. Translationally symmetric wave motions

We consider the case when the physical quantities depend on two cylindrical coordinates, r and ϕ, and not on z. Equations (4) and (5) are satisfied identically if we define the velocity and magnetic fields in terms of auxiliary scalar functions,

$$\boldsymbol{V} = \nabla \Psi(r, \phi - \omega t) \times \hat{\boldsymbol{z}} + V_z(r, \phi - \omega t)\, \hat{\boldsymbol{z}}, \tag{9}$$

$$\boldsymbol{H} = \nabla A(r, \phi - \omega t) \times \hat{\boldsymbol{z}} + H_z(r, \phi - \omega t)\, \hat{\boldsymbol{z}}, \tag{10}$$

where ω is the constant angular velocity of the wave. The contours A = constant identify the projection of the magnetic field lines, and the contours Ψ = constant identify the projection of the streamlines on the plane perpendicular to the z-direction.

Substituting (9) and (10) into the induction equation (2), we have

$$\frac{\partial A}{\partial t} + \boldsymbol{V} \cdot \nabla A = 0, \tag{11}$$

$$\frac{\partial H_z}{\partial t} + \boldsymbol{V} \cdot \nabla H_z = \boldsymbol{H} \cdot \nabla V_z. \tag{12}$$

Equations (3) and (11) indicate that both T and A move with the fluid and hence we find that the temperature is a function of A only, i.e.

$$T = T(A). \tag{13}$$

Equation (11) can also be written as

$$\frac{\partial(\Psi + \frac{1}{2} r^2 \omega, A)}{\partial(r, \phi)} = 0, \tag{14}$$

which has the general solution

$$\Psi + \tfrac{1}{2} r^2 \omega = f(A), \tag{15}$$

where f is an arbitrary function of its argument. This relation states that the projections of $\boldsymbol{V}$ and $\boldsymbol{H}$ on the plane perpendicular to the z-direction coincide, in the frame of reference ($\phi = \omega t$) moving with the wave. Making use of (15), (12) reduces to

$$\frac{\partial(V_z - f' H_z, A)}{\partial(r, \phi)} = 0, \tag{16}$$

where f' denotes the derivative of f with respect to its argument A. The general solution of (16) is

$$V_z - f'H_z = W(A), \tag{17}$$

where W is an arbitrary function of A.

Next we consider the equation of motion (1). Making use of (15) and (17), the z-component of (1) reduces to

$$(1-f'^2)\frac{\partial(H_z, A)}{\partial(r,\phi)} = 0, \tag{18}$$

which is satisfied for any form of H_z if $f' = \pm 1$ or for

$$H_z = H_z(A) \tag{19}$$

if $f' \neq \pm 1$.

Making use of (13), the curl of the r- and ϕ-components of (1) reduces to

$$\frac{\partial(\nabla^2 A - \alpha T'\Phi, A)}{\partial(r,\phi)} = \frac{\partial(\nabla^2 f, f)}{\partial(r,\phi)}. \tag{20}$$

In view of the relation $f = f(A)$, this equation finally takes the form

$$\frac{\partial[(1-f'^2)\nabla^2 A - f'f''|\nabla A|^2 - \alpha T'\Phi, A]}{\partial(r,\phi)} = 0, \tag{21}$$

which has the general solution

$$(1-f'^2)\nabla^2 A - f'f''|\nabla A|^2 - \alpha T'\Phi = M(A), \tag{22}$$

where M is an arbitrary function of its argument. It should be noted here that f' is interpreted as the Alfvén Mach number of the perturbed flow velocity perpendicular to the z-direction in terms of the perturbed Alfvén velocity perpendicular to the z-direction. Also, although the possible existence of critical points in (22) at which $f' = \pm 1$ is a real problem that can be found in astrophysical situations when the perturbed flow perpendicular to the z-direction becomes superAlfvénic in terms of the perturbed Alfvén velocity, we do not consider such a problem here and consider only for the case of $f' =$ constant ($\neq \pm 1$). It should also be noted that if $f' = \pm 1$ everywhere, then (22) reduces to $-\alpha\Phi(r) = M(A)/T'(A)$ which implies that A must be a function of r only and therefore there is no wave solution of (22) for a finite value of $\alpha T'\Phi$ when f' is a constant (being equal to ± 1). For $f' =$ constant ($\neq \pm 1$), we must solve (22), which replaces the initial set of equations (1)–(5). If we specify the arbitrary functions of $f(a)$, $M(A)$ and $T(A)$, it can be solved for A. A simple solution of (22) will be given in §4.

The scalar product of (1) and $\boldsymbol{H}$ reduces to

$$\frac{\partial\left(\frac{p}{\rho} - \Phi + \frac{1}{2}V^2 - r\omega V_\phi - UV_z + \alpha T\Phi, A\right)}{\partial(r,\phi)} = 0, \tag{23}$$

which has the general solution

$$\frac{p}{\rho} - \Phi + \tfrac{1}{2}V^2 - r\omega V_\phi - UV_z + \alpha T\Phi = \Pi(A), \tag{24}$$

where Π is an arbitrary function of A. Equation (24) provides an equation for the pressure p in terms of A.

3. Axisymmetric wave motions

Next we will seek the exact axisymmetric wave solutions for (1)–(5), in which the physical quantities are independent of the azimuthal angle ϕ. The axisymmetric velocity and magnetic fields can be expressed as the superposition of a poloidal and a toroidal component by means of two scalar functions. They are

$$\boldsymbol{V} = \nabla\chi(r, z-ct) \times \hat{\boldsymbol{\phi}}/r + V_\phi(r, z-ct)\,\hat{\boldsymbol{\phi}}, \tag{25}$$

$$\boldsymbol{H} = \nabla S(r, z-ct) \times \hat{\boldsymbol{\phi}}/r + H_\phi(r, z-ct)\,\hat{\boldsymbol{\phi}}, \tag{26}$$

where c is a constant wave velocity. The scalar functions χ and S are the stream function and the magnetic potential of the poloidal field, and the equations $\chi =$ constant and $S =$ constant define the stream and the magnetic flux surfaces, respectively.

Substituting (25) and (26) into the induction equation (2), we obtain

$$\frac{\partial S}{\partial t} + \boldsymbol{V}\cdot\nabla S = 0, \tag{27}$$

$$\frac{\partial H_\phi}{\partial t} + \boldsymbol{V}\cdot\nabla H_\phi = \boldsymbol{H}\cdot\nabla H_\phi. \tag{28}$$

Equation (27) indicates that S moves with the fluid, thus we have from (27) and (3) that

$$T = T(S). \tag{29}$$

Equation (27) can also be written as

$$\frac{\partial(\chi - \frac{1}{2}cr^2, S)}{\partial(r, z)} = 0, \tag{30}$$

which has the general solution

$$\chi - \tfrac{1}{2}cr^2 = F(S), \tag{31}$$

where F is an arbitrary function of its argument. Equation (31) relates the velocity and the magnetic field on the meridional planes. By using (31), (28) yields

$$\frac{\partial[(V_\phi - F'H_\phi)/r, S]}{\partial(r, z)} = 0, \tag{32}$$

where F' denotes the derivative of F with respect to S. The general solution of (32) is

$$V_\phi - F'H_\phi = r\Omega(S), \tag{33}$$

where Ω is an arbitrary function of S.

In a similar manner, the ϕ-component of the equation of motion (1) can be written as

$$\frac{\partial(rV_\phi, F)}{\partial(r, z)} = \frac{\partial(rH_\phi, S)}{\partial(r, z)}, \tag{34}$$

which, in view of the relation $F = F(S)$, has the general solution

$$H_\phi - F'V_\phi = L(S)/r, \tag{35}$$

with $L(S)$ an arbitrary function of S.

Equations (33) and (35) are combined to yield

$$V_\phi(1 - F'^2) = r\Omega + F'L/r, \tag{36}$$

$$H_\phi(1 - F'^2) = rF'\Omega + L/r. \tag{37}$$

If $F' = \pm 1$, then (36) and (37) reduces to

$$V_\phi = \pm H_\phi \tag{38}$$

and $\Omega = L = 0$.

The curl of the r- and z-components of (1) yields

$$(1-F'^2)\frac{\partial(\nabla^{*2}S, S)}{\partial(r,z)} - FF''\frac{\partial(|\nabla S|^2, S)}{\partial(r,z)} + \frac{1}{r}\frac{\partial}{\partial z}(V_\phi^2 - H_\phi^2) - \frac{\partial(\alpha T'\Phi, S)}{\partial(r,z)} = 0, \tag{39}$$

where
$$r\nabla^{*2}S = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial S}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 S}{\partial z^2}. \tag{40}$$

It should be noted here that F' is interpreted as the Alfvén Mach number of the perturbed poloidal flow in terms of the perturbed poloidal Alfvén velocity and that although the possible existence of critical points in (39) at which $F' = \pm 1$ is the real problem, we do not consider such a problem here and consider only the case of F' = constant ($\neq \pm 1$). It should be noted here that if $F' = \pm 1$ everywhere, then (39) reduces to $\alpha T' \partial(\Phi, S)/\partial(r,z) = 0$ which implies that S must be a function of r only, and therefore there is no wave solution of (39) for a finite value of $\alpha T'\Phi$ when F' is a constant (being equal to ± 1). Hence we will consider only F' = constant ($\neq \pm 1$). For $F' \neq \pm 1$, we find from (36) and (37) that

$$V_\phi^2 - H_\phi^2 = \frac{(r\Omega)^2 - (L/r)^2}{1-F'^2}. \tag{41}$$

Substituting (41) into (39), we obtain

$$\frac{\partial(E, S)}{\partial(r,z)} = 0, \tag{42}$$

where E is defined by

$$E = (1-F'^2)\nabla^{*2}S - \frac{|\nabla S|^2}{r^2}F'F'' + \frac{1}{2r^2}\frac{d}{dS}\frac{L^2}{1-F'^2} + \tfrac{1}{2}r^2\frac{d}{dS}\frac{\Omega^2}{1-F'^2} - \alpha T'\Phi. \tag{43}$$

Equation (42) has the general solution

$$E = E(S). \tag{44}$$

If we specify the arbitrary functions $F(S)$, $L(S)$, $\Omega(S)$, $T(S)$ and $E(S)$, the initial MHD problem of solving the set of equations (1)–(5) reduces to the mathematical problem of solving the single, elliptic partial differential equation (43) with (44) for S. Simple solutions to (43) with (44) will be presented in the next section.

The scalar product of (1) and $\boldsymbol{H}$ reduces to

$$\frac{\partial\left(\dfrac{p}{\rho} - \Phi + \tfrac{1}{2}V^2 - r\Omega V_\phi - cV_z + \alpha T\Phi, A\right)}{\partial(r,\phi)} = 0, \tag{45}$$

which has the general solution

$$\frac{p}{\rho} - \Phi + \tfrac{1}{2}V^2 - r\Omega V_\phi - cV_z + \alpha T\Phi = \Lambda(A), \tag{46}$$

where Λ is an arbitrary function of S. Equation (46) provides an equation for the pressure p in terms of A.

4. Simple solutions

The differential equations (22) and (43) with (44) each have an infinite number of mathematical solutions. In this section we treat only a few of the simpler solutions, illustrating the advantages of the equations for treating hydromagnetic wave motions in a thermally stratified cylindrical fluid.

4.1. *Translationally symmetric solutions*

First we consider (22) for translationally symmetric wave motions. As an example, if we take $f' = Q$, $M(A) = -(1-Q^2)k^2A$ and $T' = q/r_o^2\tau_o$, then (22) reduces to

$$(1-Q^2)(\nabla^2 A + k^2 A) - \epsilon \frac{\Phi}{2\pi G\rho} = 0, \tag{47}$$

where $Q\,(\neq \pm 1)$, k and q are constants, and

$$\epsilon = \frac{\alpha q(2\pi G\rho)}{r_o^2 \tau_o}. \tag{48}$$

The general solution to (47) is

$$A = \frac{\epsilon}{k^2(1-Q^2)}\left(\frac{\Phi}{2\pi G\rho} + \frac{2}{k^2}\right) + \sum_{n=0}^{\infty} [C_n J_n(kr) + \tilde{C}_n Y_n(kr)] \sin[n(\phi - \omega t) + \theta_n], \tag{49}$$

where we have used the relation (6), C_n, $\tilde{C}_n$ and θ_n are constants, $J_n(kr)$ and $Y_n(kr)$ are Bessel functions of order n of the first and second kind, and n is restricted to integral values. From (9), (10), (17), (19) and (49), we have

$$\boldsymbol{V} = r\omega\hat{\boldsymbol{\phi}} + W(A)\,\hat{\boldsymbol{z}} + Q\boldsymbol{H}, \tag{50}$$

$$T = \frac{q}{r_o^2 \tau_o} A + \text{constant}, \tag{51}$$

where
$$H_r = \frac{1}{r} \sum_{n=1}^{\infty} n[C_n J_n(kr) + \tilde{C}_n Y_n(kr)] \cos[n(\phi - \omega t) + \theta_n], \tag{52}$$

$$H_\phi = -\frac{\epsilon}{k^2(1-Q^2)} \frac{\mathrm{d}\Phi/\mathrm{d}r}{2\pi G\rho} + \sum_{n=0}^{\infty} \left[C_n \left(kJ_{n+1}(kr) - \frac{n}{r} J_n(kr)\right)\right.$$
$$\left. + \tilde{C}_n \left(kY_{n+1}(kr) - \frac{n}{r} Y_n(kr)\right)\right] \sin[n(\phi - \omega t) + \theta_n], \tag{53}$$

$$H_z = H_z(A). \tag{54}$$

The effect of the geometry is to impose a restriction on the allowed values of k. We assume that the boundaries are rigid and perfectly electrically and thermally conducting, and hence the inviscid flow satisfies

$$V_r = H_r = 0, \quad T = \text{constant} \quad \text{on} \quad r = r_i \quad \text{and} \quad r_o. \tag{55}$$

Thus, we have
$$kr_o = j_{m,l} \quad (m, l = 1, 2, 3, \ldots), \tag{56}$$

$$\tilde{C}_m = -\frac{J_m(j_{m,l})}{Y_m(j_{m,l})} C_m, \tag{57}$$

where the $j_{m,l}$ are successive positive roots of $J_m(x)\,Y_m(\gamma x) - J_m(\gamma x)\,Y_m(x) = 0$, with $\gamma = r_i/r_o$. The solution satisfying the boundary conditions is given by (A 1)–(A 9) in

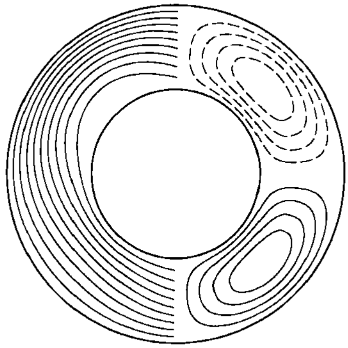

FIGURE 1. A sketch of the projection of the magnetic field lines or the streamlines on the plane perpendicular to the z-direction for the translationally symmetric wave motions. The left half shows the total (mean and perturbed) field and the right half the perturbed field.

the Appendix. The solution represents waves propagating in the ϕ-direction with the constant angular velocity ω, under the influence of helical magnetic and velocity fields $\boldsymbol{H}_0(r)$ and $\boldsymbol{V}_0(r)$. The perturbed velocity and magnetic fields are not parallel unless the function W is a constant. In figure 1 is shown a sketch of the projection of the magnetic field lines or the streamlines on the plane perpendicular to the z-direction for the simple case that $\Phi = -\pi G\rho r^2$, $C_0 = \tilde{C}_0 = 0$, $m = 2$, and $l = 1$. For the problem originally posed, we require that the mean magnetic and velocity fields must satisfy (7) and (8), respectively. Hence, we have

$$V(r) = W(A_0)+QH_z(A_0), \tag{58}$$

$$H(r) = H_z(A_0), \tag{59}$$

$$\zeta(r) = \omega-\Omega_0+Q\tau(r), \tag{60}$$

$$\tau(r) = \frac{\epsilon}{k^2(1-Q^2)}\frac{g(r)/r}{2\pi G\rho}, \tag{61}$$

where we have assumed for simplicity that $C_0 = \tilde{C}_0 = 0$, and $g(r) = -\mathrm{d}\Phi/\mathrm{d}r$ is the gravitational acceleration.

From (60) and (61) with $\zeta(r_o) = 0$ and $\tau(r_o) = \tau_o$, we find

$$\omega-\Omega_0 = -Q\tau_o, \tag{62}$$

$$\epsilon = k^2(1-Q^2)\,\tau_o\frac{2\pi G\rho}{g_o/r_o}, \tag{63}$$

where $g_o = g(r_o)$ is the gravitational acceleration at the outer boundary $r = r_o$.

From (48) and (63), we have

$$Q = \pm(1-N_1)^{\frac{1}{2}}, \tag{64}$$

where

$$N_1 = \frac{\alpha g_o q}{k^2 r_o^3 \tau_o^2}, \tag{65}$$

which measures the strength of the stratification (or the convective force). From (62) and (64), we find the angular frequency of the wave in the frame of reference rotating with the angular velocity Ω_0:

$$\omega-\Omega_0 = \mp\tau_o(1-N_1)^{\frac{1}{2}}. \tag{66}$$

On the other hand, the radial temperature gradient is given by

$$\frac{\mathrm{d}T_0}{\mathrm{d}r} = -\frac{q}{r_o}\frac{g(r)}{g_o}, \tag{67}$$

and hence we have
$$N_1 = -\frac{\alpha g_0[\mathrm{d}T_0/\mathrm{d}r]_{r=r_0}}{(kr_0\tau_0)}, \tag{68}$$
which implies that N_1 corresponds to the square of the normalized buoyancy or Brunt–Väisälä frequency.

From (67) and (68), we find that $N_1 < 0$ for $\mathrm{d}T_0/\mathrm{d}r > 0$ and $N_1 > 0$ for $\mathrm{d}T_0/\mathrm{d}r < 0$. Thus, if the fluid is stably stratified ($N_1 < 0$), then the angular velocity of the wave is increased. On the other hand, the angular velocity of the wave declines for $N_1 > 0$, reaching zero for $N_1 = 1$. For $N_1 > 1$ the field stands still and grows exponentially with the passage of time. It should be noted here that if the gravitational acceleration is given by $g(r) = 2\pi G\rho r$ which is realized when the mean density of the material inside the inner boundary $r = r_i$ is equal to that of the fluid or the inner boundary does not exist (i.e. $r_i = 0$) and $W = -QH_z$, then we have $\zeta(r) = V(r) = 0$, and hence we have $V_0 = r\Omega_0\hat{\phi}$, that is, the fluid in the stationary state rotates uniformly with the angular velocity Ω_0; otherwise the basic stationary state has necessarily a velocity shear (i.e. the differential rotation) for the existence of the wave.

4.2. *Axisymmetric solutions*

Next we consider (43) with (44) for the axisymmetric wave motion. For example, if we take $F' = Q$, $L = (1-Q^2)\beta S$, $\Omega = (1-Q^2)\Omega_0$, $E = -4\eta/\beta^2$ and $T' = q/r_0^2H_0$, then (43) reduces to
$$(1-Q^2)\left[\nabla^{*2}S+\frac{\beta^2}{r^2}S\right]-\eta\left(\frac{\Phi}{2\pi G\rho}-\frac{4}{\beta^2}\right) = 0, \tag{69}$$
where β is a constant and
$$\eta = \frac{\alpha q(2\pi G\rho)}{r_0^2H_0}. \tag{70}$$
The general solution to (69) is
$$S = \frac{\eta r^2}{\beta^2(1-Q^2)}\left(\frac{\Phi}{2\pi G\rho}-\frac{2}{\beta^2 r}\frac{\mathrm{d}\Phi/\mathrm{d}r}{2\pi G\rho}\right)+r\sum_\kappa[D_\kappa J_1(\kappa r)+\tilde{D}_\kappa Y_1(\kappa r)] \times\sin[(\beta^2-\kappa^2)^{\frac{1}{2}}(z-ct)+\beta_\kappa], \tag{71}$$
where D_κ, $\tilde{D}_\kappa$ and β_κ are real constants and $J_1(\kappa r)$ and $Y_1(\kappa r)$ are the first-order Bessel functions of the first and second kind. From (26), (37) and (71), we obtain the r-, ϕ- and z-components of $\boldsymbol{H}$:
$$H_r = -\sum_\kappa(\beta^2-\kappa^2)^{\frac{1}{2}}[D_\kappa J_1(\kappa r)+\tilde{D}_\kappa Y_1(\kappa r)]\cos[(\beta^2-\kappa^2)^{\frac{1}{2}}(z-ct)+\beta_\kappa], \tag{72}$$
$$H_\phi = Q\Omega_0 r+\beta\sum_\kappa[D_\kappa J_1(\kappa r)+\tilde{D}_\kappa Y_1(\kappa r)]\sin[(\beta^2-\kappa^2)^{\frac{1}{2}}(z-ct)+\beta_\kappa], \tag{73}$$
$$H_z = \frac{\eta}{\beta^2(1-Q^2)}\frac{2\Phi+r\,\mathrm{d}\Phi/\mathrm{d}r}{2\pi G\rho}+\sum_\kappa\kappa[D_\kappa J_0(\kappa r)+\tilde{D}_\kappa Y_0(\kappa r)]\sin[(\beta^2-\kappa^2)^{\frac{1}{2}}(z-ct)+\beta_\kappa]. \tag{74}$$
We also find from (25), (26), (31), (33) and (29) that
$$\boldsymbol{V} = (1-Q^2)\Omega_0 r\hat{\boldsymbol{\phi}}+c\hat{\boldsymbol{z}}+Q\boldsymbol{H}, \tag{75}$$
$$T = \frac{q}{r_0^2H_0}S+\text{constant}. \tag{76}$$
If we assume the same boundary conditions (55) as for the translationally symmetric wave motions, then the allowed values of κ are restricted such that $\kappa = \beta$ and
$$\kappa r_0 = j_{1,l} \quad (l = 1,2,3,\ldots), \tag{77}$$

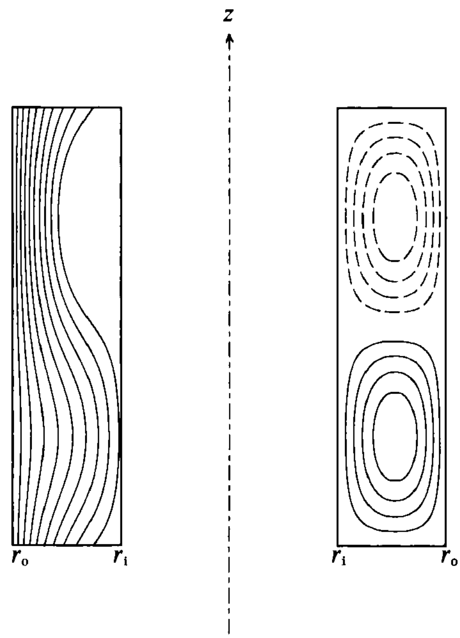

FIGURE 2. A sketch of the poloidal magnetic field lines or the poloidal streamlines for the axisymmetric wave motions. The left half shows the total (mean and perturbed) field and the right half the perturbed field.

and
$$\tilde{D}_\kappa = -\frac{J_1(j_{1,l})}{Y_1(j_{1,l})} D_\kappa, \tag{78}$$

where the $j_{1,l}$ are successive roots of $J_1(x)\,Y(\gamma x) - J_1(\gamma x)\,Y_1(x) = 0$ with $\gamma = r_i/r_o$. The solution satisfying the boundary conditions is given by (A 10)–(A 18) in the Appendix. The solution represents waves propagating in the z-direction with the constant velocity c, under the influence of helical magnetic and velocity fields $\boldsymbol{H}_0(r)$ and $\boldsymbol{V}_0(r)$. It is evident from (A 11) that the solution does not represent equipartition of energy between the magnetic and fluid variations, the factor Q setting it out of balance. Figure 2 shows a sketch of the poloidal magnetic field lines or the poloidal streamlines for the simple case where $\Phi = -\pi G\rho r^2$, $D_\beta = \tilde{D}_\beta = 0$, and $m = 1$. If we rotate these lines with respect to the z-axis, we can get the stream or the magnetic flux surfaces of the poloidal field. For the problem originally posed, we must require that the mean magnetic and velocity fields must satisfy (6) and (7), respectively. Thus, we have

$$V(r) = c + QH(r), \tag{79}$$

$$H(r) = \frac{\eta}{\beta^2(1-Q^2)} \frac{2\Phi + r\,\mathrm{d}\Phi/\mathrm{d}r}{2\pi G\rho}, \tag{80}$$

$$\xi(r) = 0, \quad \tau(r) = \tau_o = Q\Omega_0, \tag{81}$$

where we have assumed for simplicity that $D_\beta = \tilde{D}_\beta = 0$. From (79) and (80) with $V(r_o) = 0$ and $H(r_o) = H_o$, we find

$$c = -QH_o, \tag{82}$$

$$\eta = \beta^2(1-Q^2)\,H_o \frac{2\pi G\rho}{[2\Phi + r\,\mathrm{d}\Phi/\mathrm{d}r]_{r=r_o}}. \tag{83}$$

From (70) and (83), we obtain

$$Q = \pm(1-N_2)^{\frac{1}{2}}, \tag{84}$$

where

$$N_2 = \frac{\alpha q}{\beta^2 r_0^2 H_0^2}\left[2\Phi + r\frac{\mathrm{d}\Phi}{\mathrm{d}r}\right]_{r=r_0}, \tag{85}$$

which measures the strength of the stratification (or the convective force). From (82) and (84) we find

$$c = \mp(1-N_2)^{\frac{1}{2}}H_0. \tag{86}$$

From (A 12) and (A 14) with $D_\beta = \tilde{D}_\beta = 0$, we find the temperature gradient:

$$\frac{\mathrm{d}T_0}{\mathrm{d}r} = \frac{q}{r_0^2}\frac{r(2\Phi + r\,\mathrm{d}\phi/\mathrm{d}r)}{[2\Phi + r\,\mathrm{d}\Phi/\mathrm{d}r]_{r=r_0}}. \tag{87}$$

From (85) and (87), we find that $N_2 < 0$ for $\mathrm{d}T_0/\mathrm{d}r > 0$ and $N_2 > 0$ for $\mathrm{d}T_0/\mathrm{d}r < 0$ since $2\Phi + r\,\mathrm{d}\Phi/\mathrm{d}r$ is negative. Thus, if the fluid is stably stratified ($N_2 < 0$), then the wave velocity is increased. On the other hand, the wave velocity declines for $N_2 > 0$, reaching zero for $N_2 = 1$. For $N_2 > 1$ the field stands still and grows exponentially with the passage of time. It should be noted here that even if the gravitational potential is given by $\Phi(r) = -\pi G\rho r^2$, $H(r)$ and $V(r)$ are finite and depend on r for a finite value of c and therefore the basic stationary state has necessarily an axial shear flow for the existence of this type of wave solution.

As another example, if we take $F' = Q$, $L = 0$, $\Omega = (1-Q^2)\,\Omega_0$, $E = -4(1-Q^2)\,\nu^2 S$, and $T' = q/r_0 H_0$, then (43) reduces to

$$(1-Q^2)(\nabla^{*2}S + 4\nu^2 S) - \eta\frac{\Phi}{2\pi G\rho} = 0, \tag{88}$$

where ν is a real constant. The general solution to (88) is

$$S = -\frac{\eta}{8\nu^2(1-Q^2)}r^2 + \sum_\mu\left[K_\mu F_0\left(\frac{\mu^2}{8\nu^2}, \nu^2 r^2\right) + \tilde{K}_\mu G_0\left(\frac{\mu^2}{8\nu^2}, \nu^2 r^2\right)\right]\sin[\mu(z-ct)+\sigma_\mu], \tag{89}$$

where K_μ, $\tilde{K}_\mu$ and σ_μ are real constants, F_0 and G_0 are the zeroth-order regular and irregular (logarithmic) Coulomb wave functions (Abramowitz & Stegun 1965). Here we have taken the gravitational potential to be

$$\Phi = -(g_0/2r_0)\,r^2 \quad \text{with} \quad g_0 = 2\pi G\rho r_0, \tag{90}$$

which is realized when the mean density of the material inside the inner boundary $r = r_i$ is equal to that of the fluid or $r_i = 0$. If we add a term proportional to $\ln r$ to the gravitational potential, then we cannot find the solution of (88). From (26) and (89), we obtain

$$H_r = -\sum_\mu \frac{\mu}{r}\left[K_\mu F_0\left(\frac{\mu^2}{8\nu^2}, \nu^2 r^2\right) + \tilde{K}_\mu G_0\left(\frac{\mu^2}{8\nu^2}, \nu^2 r^2\right)\right]\cos[\mu(z-ct)+\sigma_\mu], \tag{91}$$

$$H_\phi = Q\Omega_0 r, \tag{92}$$

$$H_z = -\frac{\eta}{4\nu^2(1-Q^2)} + \sum_\mu\left[\frac{8+\mu^2 r^2}{4r^2}\left(K_\mu F_0\left(\frac{\mu^2}{8\nu^2}, \nu^2 r^2\right) + \tilde{K}_\mu G_0\left(\frac{\mu^2}{8\nu^2}, \nu^2 r^2\right)\right)\right.$$
$$\left. - 2\nu^2\left(1 + \left(\frac{\mu^2}{8\nu^2}\right)^2\right)^{\frac{1}{2}}\left(K_\mu F_1\left(\frac{\mu^2}{8\nu^2}, \nu^2 r^2\right) + \tilde{K}_\mu G_1\left(\frac{\mu^2}{8\nu^2}, \nu^2 r^2\right)\right)\right]\sin[\mu(z-ct)+\sigma_\mu], \tag{93}$$

where F_1 and G_1 are the first-order regular and irregular Coulomb wave functions. We also find that $\boldsymbol{V}$ and $\boldsymbol{H}$ are related by (75) and the temperature T is expressed in terms of S by (76).

If we impose the same boundary conditions (55) as for the translationally symmetric wave motions, then the allowed values of μ are restricted such that $\mu = 0$ and

$$F_0\left(\frac{\mu^2}{8\nu^2}, \nu^2 r_o^2\right) G_0\left(\frac{\mu^2}{8\nu^2}, \nu^2 r_i^2\right) - F_0\left(\frac{\mu^2}{8\nu^2}\nu^2 r_i^2\right) G_0\left(\frac{\mu^2}{8\nu^2}, \nu^2 r_o^2\right) = 0, \tag{94}$$

with
$$\tilde{K}_\mu = -\frac{F_0(\mu^2/8\nu^2, \nu^2 r_o^2)}{G_0(\mu^2/8\nu^2, \nu^2 r_o^2)} K_\mu. \tag{95}$$

The solution satisfying the boundary conditions is given by (A 19)–(A 23) with (A 10)–(A 13) in the Appendix. This solution represents waves propagating in the z-direction with the constant velocity c, under the influence of helical magnetic and velocity fields $\boldsymbol{H}_0(r)$ and $\boldsymbol{V}_0(r)$, and the non-uniform temperature $T_0(r)$. Both the perturbed velocity and magnetic fields are perpendicular to the ϕ-direction. For the problem originally posed, we require that $\boldsymbol{H}_0(r)$ and $\boldsymbol{V}_0(r)$ must satisfy (6) and (7), respectively. Hence we have

$$V(r) = c + QH_o = 0, \tag{96}$$

$$H_o = \frac{-\eta}{4\nu^2(1-Q^2)}, \tag{97}$$

$$\zeta(r) = 0, \quad \tau(r) = \tau_o = Q\Omega_0, \tag{98}$$

where we have assumed for simplicity that $K_0 = \tilde{K}_0 = 0$. From (70) and (97), we have

$$Q = \pm(1-N_3)^{\frac{1}{2}}, \tag{99}$$

where
$$N_3 = \frac{-\alpha g_o q}{4\nu^2 r_o^3 H_0^2}, \tag{100}$$

which measures the strength of the stratification (or the convective forces). From (96) and (99), we find the wave velocity

$$c = \mp H_0(1-N_3)^{\frac{1}{2}}. \tag{101}$$

On the other hand, the radial temperature gradient is given by

$$\frac{\mathrm{d}T_0}{\mathrm{d}r} = \frac{q}{r_o^2} r. \tag{102}$$

From (100) and (102) we find that $N_3 < 0$ for $\mathrm{d}T_0/\mathrm{d}r > 0$ and $N_3 > 0$ for $\mathrm{d}T_0/\mathrm{d}r < 0$. It should be noted here that although the qualitative properties of the wave for the sign of the radial temperature gradient $\mathrm{d}T_0/\mathrm{d}r$ are similar to those of the previous example, this solution represents waves propagating in the z-direction under the influence of a helical magnetic field and mean zonal flow without a shear, thus differing from the previous example.

5. Conclusions

In this paper we have presented the reduction of the vector MHD equations (in the Boussinesq approximation) in the presence of convective forces in cylindrical coordinates into a scalar, elliptic, partial differential equation of the magnetic potential for the translationally symmetric wave motions and for the axisymmetric wave motions. We have also obtained simple solutions to the reduced equations for a thermally stratified fluid in an infinite cylindrical annulus with perfectly conducting boundaries that represent waves propagating in the azimuthal and axial directions, under the influence of helical magnetic and velocity fields whose strength varies with radius. It has been shown that if the fluid is stably stratified then the wave velocity is increased as the strength of stratification is increased; and that if the sense of stratification is changed then the wave velocity declines, reaching zero with the strength of convective force, and if the strength of convective force becomes greater than a critical value then the fields grows exponentially with the passage of time. We have also shown that, except for a specific case, the waves can exist when the fluid in the stationary state has a velocity shear. It should be emphasized here that we have constructed exact solutions of the nonlinear problem by taking $f' = Q$ for the translationally symmetric wave motions or $F' = Q$ for the axially symmetric wave motions, Q being a constant, because the equation is then exactly linearized. But no superposition of solutions of that kind with different Q can be an exact solution.

A generalization of the reduction presented here to the helically symmetric wave motions will be considered in another paper of this series. Note that the helical symmetry for a stationary MHD flow was treated by Tsinganos (1982).

Appendix

If we decompose A, $\boldsymbol{V}$, $\boldsymbol{H}$ and T into stationary and fluctuating parts, i.e. $A = A_0 + a$, $\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{v}$, $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{h}$, and $T = T_0 + \Theta$, the solution for translationally symmetric wave motions satisfying the boundary conditions reduces to

$$\boldsymbol{V}_0 = r\omega\hat{\boldsymbol{\phi}} + W(A_0)\,\hat{\boldsymbol{z}} + Q\boldsymbol{H}_0, \tag{A 1}$$

$$\boldsymbol{v} = [W(A_0+a) - W(A_0)]\,\hat{\boldsymbol{z}} + Q\boldsymbol{h}, \tag{A 2}$$

$$T_0 = q/(r_0^2\tau_0)\,A_0 + \text{constant}, \tag{A 3}$$

$$\Theta = q/(r_0^2\tau_0)\,a, \tag{A 4}$$

where
$$A_0 = \frac{\epsilon}{k^2(1-Q^2)}\left(\frac{\Phi}{2\pi G\rho} + \frac{2}{k^2}\right) + C_0 J_0\left(j_{m,l}\frac{r}{r_0}\right) + \tilde{C}_0 Y_0\left(j_{m,l}\frac{r}{r_0}\right), \tag{A 5}$$

$$a = a_{m,l}(r)\sin[m(\phi - \omega t) + \theta_m], \tag{A 6}$$

$$\boldsymbol{H}_0 = \left[-\frac{\epsilon}{k^2(1-Q^2)}\frac{\mathrm{d}\Phi/\mathrm{d}r}{2\pi G\rho} + \frac{j_{m,l}}{r_0}\left(C_0 J_0\left(j_{m,l}\frac{r}{r_0}\right) + \tilde{C}_0 Y_0\left(j_{m,l}\frac{r}{r_0}\right)\right)\right]\hat{\boldsymbol{\phi}} + H_z(A_0)\,\hat{\boldsymbol{z}}, \tag{A 7}$$

$$\boldsymbol{h} = m a_{m,l}(r)\cos[m(\phi-\omega t)+\theta_m]\,\hat{\boldsymbol{r}} - \frac{\mathrm{d}a_{m,l}(r)}{\mathrm{d}r}\sin[m(\phi-\omega t)+\theta_m]\,\hat{\boldsymbol{\phi}} + [H_z(A_0+a) - H_z(A_0]\,\hat{\boldsymbol{z}}, \tag{A 8}$$

$$a_{m,l}(r) = C_m\left[J_m\left(j_{m,l}\frac{r}{r_0}\right) - \frac{J_m(j_{m,l})}{Y_m(j_{m,l})}Y_m\left(j_{m,l}\frac{r}{r_0}\right)\right]. \tag{A 9}$$

If we decompose S, $\boldsymbol{V}$, $\boldsymbol{H}$ and T into stationary and fluctuating parts, i.e., $S = S_0 + s$, $\boldsymbol{V} = \boldsymbol{V}_0 + \boldsymbol{v}$, $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{h}$ and $T = T_0 + \Theta$, the solution of the first example for the axisymmetric wave motions satisfying the boundary conditions becomes

$$\boldsymbol{V}_0 = (1-Q^2)\,\Omega_0 r\hat{\boldsymbol{\phi}} + c\hat{\boldsymbol{z}} + Q\boldsymbol{H}_0, \tag{A 10}$$

$$\boldsymbol{v} = Q\boldsymbol{h}, \tag{A 11}$$

$$T_0 = q/(r_0^2 H_0)\,S_0 + \text{constant}, \tag{A 12}$$

$$\Theta = q/(r_0^2 H_0)\,s, \tag{A 13}$$

where $$S_0 = \frac{\eta r^2}{\beta^2(1-Q^2)}\left(\frac{\Phi}{2\pi G\rho} - \frac{2}{\beta^2 r}\frac{\mathrm{d}\Phi/\mathrm{d}r}{2\pi G\rho} - \frac{2}{\beta^2}\right) + D_\beta r J_1(\beta r) + \tilde{D}_\beta r Y_1(\beta r), \tag{A 14}$$

$$s = \sum_{n=1}^{m} s_n(r)\sin\left[\frac{\gamma_n(z-ct)}{r_0} + \delta_n\right], \tag{A 15}$$

$$\boldsymbol{H}_0 = [Q\Omega_0 r + \beta(D_\beta J_1(\beta r) + \tilde{D}_\beta Y_1(\beta r))]\hat{\boldsymbol{\phi}} + \left[\frac{\eta}{\beta^2(1-Q^2)}\frac{2\Phi + r\,\mathrm{d}\Phi/\mathrm{d}r}{2\pi G\rho} + \beta(D_\beta J_0(\beta r) + \tilde{D}_\beta Y_0(\beta r))\right]\hat{\boldsymbol{z}}, \tag{A 16}$$

$$\boldsymbol{h} = -\sum_{n=1}^{m}\frac{\gamma_n}{r_0}\frac{s_n(r)}{r}\cos\left[\frac{\gamma_n(z-ct)}{r_0} + \delta_n\right]\hat{\boldsymbol{r}} + \frac{\beta}{r}\sum_{n=1}^{m} s_n(r)\sin\left[\frac{\gamma_n(z-ct)}{r_0} + \delta_n\right]\hat{\boldsymbol{\phi}} + \frac{1}{r}\sum_{n=1}^{m}\frac{\mathrm{d}s_n(r)}{\mathrm{d}r}\sin\left[\frac{\gamma_n(z-ct)}{r_0} + \delta_n\right]\hat{\boldsymbol{z}}, \tag{A 17}$$

$$s_n(r) = \bar{D}_n r\left[J_1\left(j_{1,n}\frac{r}{r_0}\right) - \frac{J_1(j_{1,n})}{Y_1(j_{1,n})}Y_1\left(j_{1,n}\frac{r}{r_0}\right)\right], \tag{A 18}$$

where $\gamma_n = (\beta^2 r_0^2 - j_{1,n}^2)^{\frac{1}{2}}$, $\bar{D}_n$ and δ_n are constants, and m is a positive integer satisfying $|\beta r_0| > j_{1,m}$. For the other example of the axisymmetric wave motions, the solution satisfying the boundary conditions is given by

$$S_0 = -\frac{\eta}{8\nu^2(1-Q^2)}r^2 + K_0\sin(\nu^2 r^2) + \tilde{K}_0\cos(\nu^2 r^2), \tag{A 19}$$

$$s = \sum_\mu s_\mu(r)\sin[\mu(z-ct) + \sigma_\mu], \tag{A 20}$$

$$\boldsymbol{H}_0 = Q\Omega_0 r\hat{\boldsymbol{\phi}} + \left[\frac{-\eta}{4\nu^2(1-Q^2)} + K_0\cos(\nu^2 r^2) - \tilde{K}_0\sin(\nu^2 r^2)\right]\hat{\boldsymbol{z}}, \tag{A 21}$$

$$\boldsymbol{h} = -\sum_\mu \frac{\mu}{r}s_\mu(r)\cos[\mu(z-ct) + \sigma_\mu]\hat{\boldsymbol{r}} + \sum_\mu \frac{1}{r}\frac{\mathrm{d}s_\mu(r)}{\mathrm{d}r}\sin[\mu(z-ct) + \sigma_\mu]\hat{\boldsymbol{z}}, \tag{A 22}$$

$$s_\mu(r) = K_\mu\left[F_0\left(\frac{\mu^2}{8\nu^2}, \nu^2 r^2\right) - \frac{F_0(\mu^2/8\nu^2, \nu^2 r_0^2)}{G_0(\mu^2/8\nu^2, \nu^2 r_0^2)}G_0\left(\frac{\mu^2}{8\nu^2}, \nu^2 r^2\right)\right], \tag{A 23}$$

where $\sum_\mu$ means the summation over all μ satisfying (94). The mean and perturbed velocities $\boldsymbol{V}_0$ and $\boldsymbol{v}$ and the mean and perturbed temperatures T_0 and Θ are given by (A 10)–(A 13).

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